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Rook theory and cycle-counting permutation statistics

Fred Butler [1]

Department of Mathematics, West Virginia University, 320 Armstrong Hall, Morgantown, WV 26506-6310, USA

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Abstract

A statistic is found to combinatorially generate the cycle-counting q-hit numbers, defined algebraically by Haglund [Adv. in Appl. Math. 17 (1996) 408–459]. We then define the notion of a cycle-Mahonian pair of statistics (generalizing that of a Mahonian statistic), and show that our newly discovered statistic is part of such a pair. Finally, we note a second example of a cycle-Mahonian pair of statistics which leads us to define the stronger property of being a cycle-Euler–Mahonian pair.

Keywords: Rook theory; q-analog; Cycle-counting; Permutation statistics; Major index; Mahonian statistic; Euler–Mahonian statistic

1. Introduction

In classical rook theory, a *board* is a subset of the $n \times n$ square board (which we shall call SQ_n) depicted in Fig. 1. Let $B(b_1, \ldots, b_n)$ denote the board $B \subseteq SQ_n$ consisting of all squares $\{(i, j) \mid j \leqslant b_i\}$. For example, $B(2, 1, 3)$ is pictured in Fig. 2. When we also have $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{n-1} \leqslant b_n$, we call $B(b_1, \ldots, b_n)$ a *Ferrers board*. Another way to specify a Ferrers board, which we will use frequently here, is to give the step heights and depths. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$ is shown in Fig. 3. A q-analogue of rook theory,

E-mail address: fbutler@math.upenn.edu.

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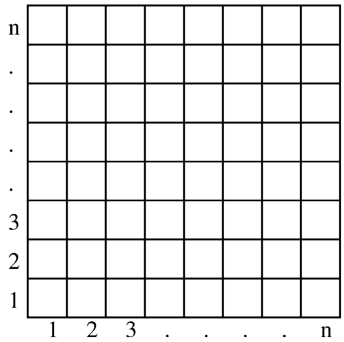

Fig. 1. The $n \times n$ square board SQ_n.

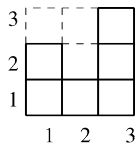

Fig. 2. The board $B(2, 1, 3) \subseteq SQ_3$.

first introduced in [5], focuses on Ferrers boards. In this paper we will concentrate on *regular Ferrers boards*, which are Ferrers boards with the additional property that $b_i \geqslant i$ for $1 \leqslant i \leqslant n$ (or equivalently, $h_1 \geqslant d_1$, $h_1 + h_2 \geqslant d_1 + d_2$, $\ldots$, $h_1 + h_2 + \cdots + h_t \geqslant d_1 + d_2 + \cdots + d_t$ as defined in [7]).

A *rook placement* on a board $B \subseteq SQ_n$ is a subset of squares of B such that no two of these squares lie in the same row or the same column. As the name suggests, these squares represent positions on an $n \times n$ chess board where non-attacking rooks can be placed. We denote the set of all placements of k non-attacking rooks on B by $\mathcal{R}_k(B)$, and the number of ways of placing k non-attacking rooks on B by $r_k(B)$, called the *kth rook number* of B. Note that $r_k(B) = |\mathcal{R}_k(B)|$. The set of all placements of n non-attacking rooks on SQ_n such

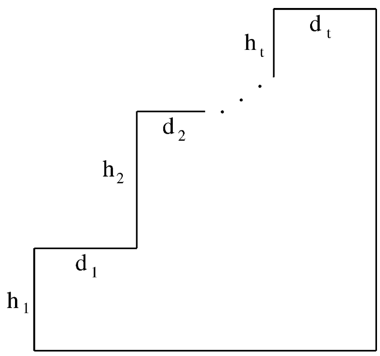

Fig. 3. The Ferrers board $B(h_1, d_1; \ldots; h_t, d_t)$.

that exactly k of the rooks lie on B is denoted $\mathcal{H}_{n,k}(B)$. The number of such placements (that is, $|\mathcal{H}_{n,k}(B)|$), written $h_{n,k}(B)$, is called the *kth hit number* of B relative to SQ_n.

Given a placement P of rooks on a Ferrers board $B \subseteq SQ_n$ we can define the following three statistics for P. First, if we let each rook cancel all squares to the right in its row and below in its column, then as in [5] we can define $\mathrm{inv}(P)$ to be the number of uncancelled squares of B. That is, $\mathrm{inv}(P)$ is the number of squares on B which are not cancelled by the above scheme and also do not contain a rook from P.

Next, it is possible to associate to a rook placement P on a board $B \subseteq SQ_n$ a simple directed graph G_P on n vertices, a fact first noted in [6] (see also [1] and [2]). A rook from P occupies the square (i, j) if and only if there is an edge from i to j in G_P. We see that G_P is a directed graph on n vertices with some cycles and some directed paths (where vertices with no incident edges count as a directed path of length one). Hence we can define $\mathrm{cyc}(P)$ to be the number of cycles in G_P. Note that the definitions of G_P and $\mathrm{cyc}(P)$ make sense even if B is not a Ferrers board.

The final statistic depends on the following fact. Let P be any placement of j non-attacking rooks in columns 1 through $i-1$ of a Ferrers board $B = B(b_1, \ldots, b_n)$ (where $j \leqslant i-1$), and let G_P be the associated directed graph as above. If $b_i \geqslant i$ then there is exactly one square on B in column i such that placing a rook on this square will complete a new cycle in G_P, whereas if $b_i < i$ then there is no such square on B. This fact can be seen by the following argument.

If $b_i \geqslant i$, then either there is a directed path in G_P which ends with i or there is not. If there is such a directed path then it must begin with some $k < i$, and (i, k) is the unique square in column i on which placing a rook will complete a cycle in G_P. The square (i, k) lies on B because $k < i \leqslant b_i$. If there is no such directed path, then placing a rook on (i, i) will complete a cycle in G_P. The square (i, i) clearly lies on B because $b_i \geqslant i$. Thus we see in this case there is always a unique square on B in column i which will complete a cycle.

If $b_i < i$ and we place a rook on B in column i on square (i, k), we know that $k \leqslant b_i < i$. In order for the placement of a rook on (i, k) to complete a cycle in G_P, we need a directed path in G_P beginning with k and ending with i. In particular, we must have a rook on the square (ℓ, i) for some $\ell < i$. However, the square (ℓ, i) cannot possibly lie on B because B is a Ferrers board, and hence $\ell < i$ implies that $b_\ell \leqslant b_i < i$. Thus in this case there is no square in column i of the Ferrers board B which will complete a cycle.

Now suppose P is a placement of some number of rooks on the Ferrers board $B = B(b_1, \ldots, b_n)$. We can then define, for those i with $b_i \geqslant i$, $s_i(P)$ to be the unique square which, considering only the rooks from P in columns 1 through $i-1$ of B, completes a cycle. Then let $E(P)$ be the number of i such that $b_i \geqslant i$ and there is no rook from P in column i on or above square $s_i(P)$.

Garsia and Remmel in [5] used the statistic inv to define the *kth q-rook number* of a Ferrers board $B = B(b_1, \ldots, b_n) \subseteq SQ_n$ by

$$R_k(q, B) = \sum_{P \in \mathcal{R}_k(B)} q^{\mathrm{inv}(P)}$$

and the *q-hit numbers* via the equation

$$\sum_{k=0}^{n} A_{n,k}(q,B)z^k = \sum_{k=0}^{n} R_{n-k}(q,B)[k]!\, z^k \prod_{i=k+1}^{n} \left(1 - zq^i\right),$$

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [n][n-1]\cdots[2][1]$ for $n \in \mathbb{N}$.

Note that both Dworkin [3] and Haglund [8] gave descriptions of different statistics such that

$$A_{n,k}(q,B) = \sum_{P \in \mathcal{H}_{n,n-k}(B)} q^{\mathrm{stat}(P)}.$$

Haglund's statistic, which we will denote $s_{B,h}(P)$, is given by the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme.

1. Each rook cancels all squares to the right in its row.
2. Each rook on B cancels all squares above it in its column.
3. Each rook off B cancels all squares below it but off B in its column.

Thus if $B \subseteq SQ_6$ is enclosed by the solid lines in Fig. 4 and P is the placement shown, then $s_{B,h}(P) = 8$.

If we let $[y] = (1 - q^y)/(1 - q)$ for any real number y (generalizing the previous definition of $[n]$ for $n \in \mathbb{N}$), we can now define the *kth cycle-counting q-rook number* of B via

$$R_k(y,q,B) = \sum_{P \in \mathcal{R}_k(B)} [y]^{\mathrm{cyc}(P)} q^{\mathrm{inv}(P)+(y-1)E(P)}$$

as in [4], and the *cycle-counting q-hit numbers* via the equation

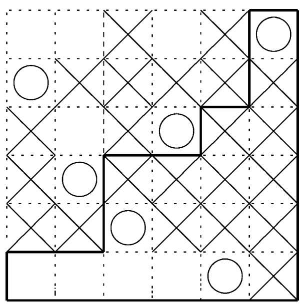

Fig. 4. A placement P on $B = B(1,1,3,3,4,6) \subseteq SQ_6$ with $s_{B,h}(P) = 8$.

$$\sum_{k=0}^{n} R_{n-k}(y,q,B)[y][y+1]\cdots[y+k-1]z^k \prod_{i=k+1}^{n}\left(1-zq^{y+i-1}\right)$$
$$=\sum_{k=0}^{n} A_{n,k}(y,q,B)z^k.$$

What we refer to as $A_{n,k}(y,q,B)$ is the same as $A_k(x,y,B)$ as defined in [7] for the case $x=y$. Note that the $R_k(y,q,B)$ generalize both the q-rook numbers of Garsia and Remmel [5] and the cycle-counting rook numbers discussed in [1,2] and [7], and the $A_{n,k}(y,q,B)$ analogously generalize both the q-hit numbers and the cycle-counting hit numbers.

In Section 2 of this paper, we find an expression for the $A_{n,k}(y,q,B)$ in terms of the ordinary q-hit numbers of a specific larger board, when $y \in \mathbb{N}$. In Section 3 we define a mapping which takes a placement on the larger board and maps it to a placement on the original board B. We will exploit Haglund's statistic for combinatorially generating the q-hit numbers to prove several useful lemmas about this mapping. In Section 4 we present the main result of this paper, a statistic which combinatorially generates the $A_{n,k}(y,q,B)$. Finally, in Section 5 we apply this statistic to give some new results on permutation statistics involving cycle-counting.

2. $A_{n,k}(y,q,B)$ when $y \in \mathbb{N}$

If $B=B(h_1,d_1;\ldots;h_t,d_t)=B(b_1,\ldots,b_n)$ is a Ferrers board then we define, for $1 \leqslant p \leqslant t$, the Ferrers board

$$B-h_p-d_p := B(h_1,d_1;\ldots;h_p-1,d_p-1;\ldots;h_t,d_t).$$

Also, let us denote the number of squares of B by $\mathrm{Area}(B)$, so $\mathrm{Area}(B)=b_1+\cdots+b_n$. Finally we define, for $m \in \mathbb{N}$,

$$B_m = B(h_1+m-1,d_1;\ldots;h_t,d_t+m-1).$$

If B is a regular Ferrers board (and hence $b_n=n$), then B_m is regular with $n+m-1$ columns, of heights

$$b_1+m-1,\quad b_2+m-1,\quad \ldots,\quad b_n+m-1,\quad \underbrace{n+m-1,\ \ldots,\ n+m-1}_{m-1}.$$

Note since at least the last m columns of $B_m \subseteq SQ_{n+m-1}$ for any regular Ferrers board B have height $n+m-1$, any rooks in the last m columns of SQ_{n+m-1} must be on B_m. Thus in particular any placement of $n+m-1$ rooks on SQ_{m+n-1} must have at least m rooks on B_m, so $\mathcal{H}_{n+m-1,k}(B_m)=\emptyset$ for $0 \leqslant k \leqslant m-1$.

We use the following two lemmas to prove the main proposition of this section.

Lemma 2.1. *For B a regular Ferrers board, $m \in \mathbb{N}$ and B_m as defined above,*

$$A_{n,0}(m,q,B) = A_{n+m-1,0}(q,B_m)/[m-1]!.$$

Proof. By definition $A_{n,0}(y,q,B) = R_n(y,q,B)$, and by (47) of [7] with $x = 0$,

$$R_n(y,q,B) = \prod_{i=1}^{n}[b_i - i + y] = \prod_{i=1}^{n}\bigl[(b_i + y - 1) - i + 1\bigr].$$

Hence $A_{n,0}(m,q,B) = \prod_{i=1}^{n}[(b_i + m - 1) - i + 1]$. By the definition of Haglund's statistic for generating the q-hit numbers,

$$\begin{aligned}
A_{n+m-1,0}(q,B_m) &= [b_1 + m - 1]\bigl[(b_2 + m - 1) - 1\bigr]\cdots\bigl[(b_n + m - 1) - n + 1\bigr] \\
&\quad \times\bigl[(n + m - 1) - n\bigr]\bigl[(n + m - 1) - n - 1\bigr] \times \cdots \\
&\quad \times\bigl[(n + m - 1) - n - m + 2\bigr] \\
&= \prod_{i=1}^{n}\bigl[(b_i + m - 1) - i + 1\bigr] \times [m-1]!,
\end{aligned}$$

and the lemma follows. □

Lemma 2.2. *For any regular Ferrers board $B = B(h_1, d_1; \ldots; h_t, d_t)$ we have that*

$$\begin{aligned}
A_{n,k}(y,q,B) &= [y + k + d_t - 1]A_{n-1,k}(y,q,B - h_t - d_t) \\
&\quad + q^{y+k+d_t-2}[n - k - d_t + 1]A_{n-1,k-1}(y,q,B - h_t - d_t)
\end{aligned}$$

for $0 < k \leqslant n$.

Proof. Let $x = y$ and $p = t$ in Lemma 5.7 of [7]. □

The next proposition is integral to proving the main result of the paper in Section 4.

Proposition 2.3. *For any regular Ferrers board B and $m \in \mathbb{N}$, we have that*

$$A_{n,k}(m,q,B) = \frac{A_{n+m-1,k}(q,B_m)}{[m-1]!}$$

for $0 \leqslant k \leqslant n$.

Proof. We will prove this proposition by induction on $\mathrm{Area}(B)$. When $\mathrm{Area}(B) = 1$ the only regular Ferrers board is the 1×1 square SQ_1, and an easy calculation shows that $A_{1,0}(m,q,SQ_1)=[m]$ and $A_{1,k}(m,q,SQ_1)=0$ for all $k>0$. By the definition of $s_{B_m,h}(P)$ given in Section 1, $A_{1+m-1,0}(q,B_m)=A_{m,0}(q,SQ_m)=[m]!$ and $A_{m,k}(q,SQ_m)=0$ for $k>0$, so the proposition holds in this case.

Now assume the proposition holds for all regular Ferrers boards of Area $<A$, and suppose $B=B(h_1,d_1;\dots;h_t,d_t)=B(b_1,\dots,b_n)$ is such that Area$(B)=A$. By Lemma 2.1, we have that $A_{n,0}(m,q,B)=A_{0,n+m-1}(q,B_m)/[m-1]!$. Then by Lemma 2.2 when $y=m$, we have for $k>0$ that $A_{n,k}(m,q,B)$ equals

$$[m+k+d_t-1]A_{n-1,k}(m,q,B-h_t-d_t)\\ +q^{m+k+d_t-2}[n-k-d_t+1]A_{n-1,k-1}(m,q,B-h_t-d_t),$$

which is

$$\big[k+(d_t+m-1)\big]A_{n-1,k}(m,q,B-h_t-d_t)\\ +q^{k+(d_t+m-1)-1}\big[(n+m-1)-(d_t+m-1)-k+1\big]\\ \times A_{n-1,k-1}(m,q,B-h_t-d_t). \tag{1}$$

By induction, $A_{n-1,k}(m,q,B-h_t-d_t)=A_{(n-1)+m-1,k}(q,(B-h_t-d_t)_m)/[m-1]!$, which equals

$$A_{(n-1)+m-1,k}\big(q,B(h_1+m-1,d_1;\dots;h_t-1,d_t-1+m-1)\big)/[m-1]!,$$

and $A_{n-1,k-1}(m,q,B-h_t-d_t)$ is

$$A_{(n-1)+m-1,k-1}\big(q,B(h_1+m-1,d_1;\dots;h_t-1,d_t-1+m-1)\big)/[m-1]!.$$

Thus (1) is equal to

$$\big[k+(d_t+m-1)\big]\\ \times A_{(n-1)+m-1,k}\big(q,B(h_1+m-1,d_1;\dots;h_t-1,d_t-1+m-1)\big)/[m-1]!\\ +q^{k+(d_t+m-1)-1}\big[(n+m-1)-(d_t+m-1)-k+1\big]\\ \times A_{(n-1)+m-1,k-1}\big(q,B(h_1+m-1,d_1;\dots;h_t-1,d_t-1+m-1)\big)/[m-1]!$$

which is

$$\frac{1}{[m-1]!}\Big\{\big[k+(d_t+m-1)\big]\\ \times A_{(n-1)+m-1,k}\big(q,B(h_1+m-1,d_1;\dots;h_t-1,d_t-1+m-1)\big)\\ +q^{k+(d_t+m-1)-1}\big[(n+m-1)-(d_t+m-1)-k+1\big]\\ \times A_{(n-1)+m-1,k-1}\big(q,B(h_1+m-1,d_1;\dots;h_t-1,d_t-1+m-1)\big)\Big\}.$$

Now by Lemma 2.2 with $y = 1$, the above is equal to

$$\frac{1}{[m-1]!}A_{n+m-1,k}\big(q, B\big(h_1+m-1, d_1; \ldots; (h_t-1)+1, (d_t-1+m-1)+1\big)\big)$$
$$= \frac{1}{[m-1]!}A_{n+m-1,k}\big(q, B(h_1+m-1, d_1; \ldots; h_t, d_t+m-1)\big),$$

which is

$$\frac{1}{[m-1]!}A_{n+m-1,k}(q, B_m)$$

and the proposition follows. □

3. The map $\phi_{n,B,m}$ and its properties

For any Ferrers board $F \subseteq SQ_d$, let us denote $\bigcup_{i=0}^{d} \mathcal{H}_{d,i}(F)$ by $\mathcal{P}_d(F)$. Throughout this section let $B \subseteq SQ_n$ be some fixed regular Ferrers board, $B_m \subseteq SQ_{n+m-1}$ as previously defined for some $m \in \mathbb{N}$. If $P \in \mathcal{P}_{n+m-1}(B_m)$, let $r_i(P)$ denote the rook from P in the ith column of SQ_{n+m-1}, and analogously for $Q \in \mathcal{P}_n(B)$ and $r_i(Q)$.

We define a mapping $\phi_{n,B,m} : \mathcal{P}_{n+m-1}(B_m) \to \mathcal{P}_n(B)$ as follows. Suppose $P \in \mathcal{P}_{n+m-1}(B_m)$. Beginning in column 1 and proceeding from left to right one column at a time, the following occurs.

1. $r_i(P)$ is on one of the m lowest squares in column i not attacked by a rook to the left if and only if $r_i(P)$ maps to the unique square $s_i(\phi_{n,B,m}(P))$ which completes a cycle in the image of P so far. (That is, you consider the placement of rooks on $SQ_n \supseteq B$ in columns 1 through $i-1$ given by $\phi_{n,B,m}(r_1(P)), \phi_{n,B,m}(r_2(P)), \ldots, \phi_{n,B,m}(r_{i-1}(P))$, and $s_i(\phi_{n,B,m}(P))$ is the unique square in column i which would complete a cycle in this placement.)
2. Otherwise, $r_i(P)$ is on the $(m+a_i)$th square $(a_i > 0)$ in column i not attacked by a rook to the left if and only if $r_i(P)$ maps to the a_ith available square in column i of B so far *which does not complete a cycle* (that is, the a_ith available square in column i of B, not counting the square $s_i(\phi_{n,B,m}(P))$ described above).

The best way to understand this mapping is to do an example in detail. Consider the placement P of 6 rooks on the board $SQ_6 \supseteq B_3$, where $B = B(1,3,4,4) \subseteq SQ_4$. This board and placement are depicted in Fig. 5. The leftmost rook $r_1(P)$ is in the fifth available position in its column, which is also the fifth square in this column not attacked by a rook to the left (because there are no rooks to the left). Since $m = 3$ in this case (so $5 = m + 2$), $\phi_{4,B,3}(r_1(P))$ is on the second available square in column 1 of SQ_4 which does not complete a cycle. Since the square $(1,1)$ is always the cycle square in the first column, $r_1(P)$ maps to square $(1,3)$.

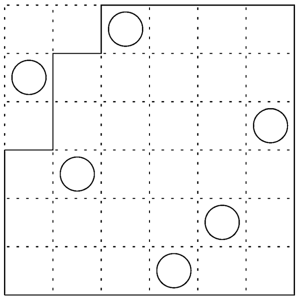

Fig. 5. A placement P of rooks on $B_m \subseteq SQ_{n+m-1}$ for $B = B(1,3,4,4)$, $m = 3$, and $n = 4$.

Now the cycle square in column 2 of B is $(2,2)$. Since $r_2(P)$ is on one of the 3 lowest squares in column 2 of SQ_6 not attacked by a rook to the left, $\phi_{4,B,3}(r_2(P))$ is on the cycle square $(2,2)$.

At this point the cycle square is $(3,1)$. Here $r_3(P)$ is on the fourth square not attacked by a rook to the left (and $4 = m + 1$), so $\phi_{4,B,3}(r_3(P))$ is on the first available square of SQ_4 which does not complete a cycle. In this case square $(3,1)$ is the cycle square, and squares $(3,2)$ and $(3,3)$ are attacked by the rooks in columns 1 and 2 of SQ_4, so the first available non-cycle square is $(3,4)$.

Finally, the cycle square in column 4 of SQ_4 is $(4,1)$. Since $r_4(P)$ in on the lowest square in its column (and hence one of the 3 lowest not attacked by a rook to the left), $\phi_{4,B,3}(r_4(P))$ is on the cycle square. The image $\phi_{4,B,3}(P)$ is depicted in Fig. 6.

The general principle behind $\phi_{n,B,m}$ is the following. Suppose you want to map a rook in column i of a placement P on $SQ_{n+m-1} \supseteq B_m$. Imagine covering columns $i+1$ through $n+m-1$ of SQ_{n+m-1}, so that only columns 1 through i can be seen. If $r_i(P)$ is on one of the m lowest available squares in column i of this "covered" board, then $r_i(P)$ maps to the square of $SQ_n \supseteq B$ which completes a cycle in the image so far. The remaining $(n+m-1)-(i-1)-m = n-i$ squares in column i of SQ_{n+m-1} are then mapped in order to the $n-(i-1)-1 = n-i$ available non-cycle squares in column i of SQ_n. Figure 7 illustrates this idea further.

Note that in the above definition of $\phi_{n,B,m}$ we ignore the rooks from a placement $P \in \mathcal{P}_{n+m-1}(B_m)$ in columns $n+1$ through $n+m-1$ of SQ_{n+m-1}. Thus for a fixed arrangement of n rooks in columns 1 through n of SQ_{n+m-1}, we see there will be $(m-1)!$ total ways to arrange the rooks in the last $m-1$ columns of SQ_{n+m-1}. Hence these $(m-1)!$ placements will all map to the same placement of n rooks on SQ_n.

We have the following lemmas.

Lemma 3.1. *$\phi_{n,B,m}$ is surjective.*

Proof. Given a placement $Q \in \mathcal{P}_n(B)$, we build a placement $P \in \mathcal{P}_{n+m-1}(B)$ from left to right. If the rook from Q in the ith column is on the square which completes a cycle, then we choose $r_i(P)$ to be on one of the m lowest available squares of SQ_{n+m-1} (so for each rook on a cycle square from Q, we will have m choices for the rook from P in the same column). If $r_i(Q)$ is on the a_ith square in its column not attacked by a rook

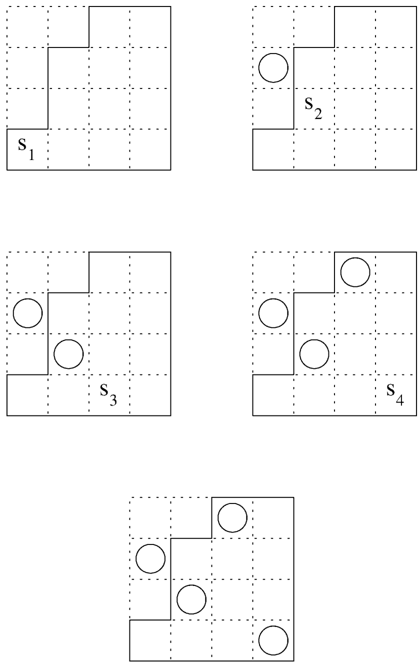

Fig. 6. The image of P from Fig. 5 under $\phi_{4,B,3}$ at each step; the cycle square in column i is denoted s_i.

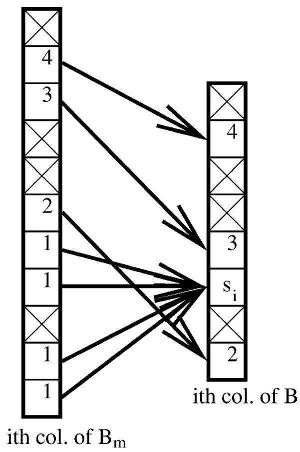

Fig. 7. The general idea behind the map $\phi_{n,B,m}$ in the ith column.

to the left and which does not complete a cycle, then $r_i(P)$ must be on the $(m + a_i)$th available square in column i of SQ_{n+m-1}. Once the rooks in columns 1 through n are determined, we choose any arrangement of rooks in columns $n+1$ through $n+m-1$ which results in a non-attacking placement. It is clear that this procedure will result in a placement $P \in \mathcal{P}_{n+m-1}(B)$, and each rook from P was chosen to ensure that $Q = \phi_{n,B,m}(P)$. $\square$

Lemma 3.2. *Let $B \subseteq SQ_n$ be a regular Ferrers board, $m \in \mathbb{N}$, $\phi_{n,B,m} : \mathcal{P}_{n+m-1}(B_m) \to \mathcal{P}_n(B)$. Let $P \in \mathcal{P}_{n+m-1}(B_m)$, and $Q = \phi_{n,B,m}(P)$. For $1 \leqslant i \leqslant n$, $r_i(P)$ is on B_m if and only if $r_i(Q)$ is on B, and $r_i(P)$ is off B_m on square $(i, j_i + m - 1)$ if and only if $r_i(Q)$ is off B on square (i, j_i).*

Proof. Fix n, B and m; the proof is by induction on i. If $i = 1$, then by definition of $\phi_{n,B,m}$ any rook on one of the m lowest squares in column 1 maps to the unique square in column 1 of B which completes a cycle, namely $(1, 1)$, and a rook on square $(1, j + m - 1)$ (for $j > 1$) maps to square $(1, j)$. Thus $r_1(P)$ is on B_m if and only if $r_1(Q)$ is on B, and $r_1(P)$ is off B_m on square $(1, j + m - 1)$ if and only if $r_1(Q)$ is off B on square $(1, j)$ as desired.

Now consider the rook $r_i(P)$ in column i of P for $i > 1$. Let k_i denote the number of rooks from P in columns 1 through $i - 1$ which can attack a square on B_m in column i; that is, k_i is the number of rooks in columns 1 through $i - 1$ of SQ_{n+m-1} which are in rows 1 through $b_i + m - 1$, where b_i denotes the height of column i of B. Then we see that there are $b_i + m - 1 - k_i$ available squares in column i of SQ_{n+m-1} which are on B_m.

By induction, any rook from P in columns 1 through $i - 1$ is on B_m if and only if this rook maps to a rook on B, and any rook is off B_m on square $(s, j_s + m - 1)$ if and only if this rook maps to a rook off B on square (s, j_s). These two facts imply that a rook in columns 1 through $i - 1$ in a row between 1 and $b_i + m - 1$ of SQ_{n+m-1} maps to a rook in columns 1 through $i - 1$ of SQ_n in a row between 1 and b_i. Thus the number of rooks in columns 1 through $i - 1$ of SQ_n from Q which can attack a square on B is also k_i, and hence there are $b_i - k_i$ available squares in column i of SQ_n which are on B.

A rook on one of the lowest m available squares in column i of SQ_{n+m-1} will map to the unique square in column i of SQ_n which completes a cycle in Q. Since B is a regular Ferrers board, this square will lie on B. Thus there are $(b_i + m - 1) - k_i - m = b_i - k_i - 1$ available squares on B_m in column i of SQ_{n+m-1} which do not map to $s_i(Q)$. On SQ_n we see that there is one square which completes a cycle in Q, and $b_i - k_i - 1$ squares which do not complete a cycle. Hence by the definition of $\phi_{n,B,m}$ the $b_i - k_i - 1$ squares on B_m which do not map to $s_i(Q)$ are in one to one correspondence with the $b_i - k_i - 1$ available squares on B in column i, so $r_i(P)$ is on B_m if and only if $r_i(Q)$ is on B.

Finally, the remaining $(n + m - 1) - (b_i + m - 1) - (i - 1 - k_i) = n - b_i - i + 1 + k_i$ available squares in column i of SQ_{n+m-1} off B_m are in one-to-one correspondence with the $n - b_i - (i - 1 - k_i) = n - b_i - i + 1 + k_i$ available squares in column i of SQ_n off B. By induction a rook on square $(s, j_s + m - 1)$ for $1 \leqslant s \leqslant i - 1$ which is off B_m maps to a rook on square (s, j_s) which is off B. Thus we see that in column i a square $(i, j_i + m - 1)$ off B_m is available if and only if the square (i, j_i) (which is off B) is available. Thus by definition of $\phi_{n,B,m}$ we see that $r_i(P)$ is off B_m on square $(i, j_i + m - 1)$ if and only if $r_i(Q)$ is off B on square (i, j_i). $\square$

Note that a corollary of Lemma 3.2 is that $\phi_{n,B,m}|_{\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)}$ is actually a map from $\mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)$ to $\mathcal{H}_{n,n-k}(B)$.

Now let us weight a placement $Q \in \mathcal{H}_{n,k}(B)$ by

$$\sum_{P \in \phi_{n,B,m}^{-1}(Q)} q^{s_{B_m,h}(P)}, \tag{2}$$

where $s_{B_m,h}(P)$ is as described in Section 1. As was earlier discussed, the rooks from some $P \in \mathcal{P}_{n+m-1}(B_m)$ in columns $n+1$ through $n+m-1$ will all lie on B_m. Thus by the definition of Haglund's statistic $s_{B_m,h}(P)$, if we fix the rooks in the first n columns and sum over all the possible $(m-1)!$ placements of non-attacking rooks in the last $m-1$ columns, we will generate $[m-1]!$.

Given a statistic stat which can be calculated for any rook placement R on a board $SQ_d \supseteq F$, we will denote by $\mathrm{stat}(R)_i$ the contribution to $\mathrm{stat}(R)$ coming from the ith column of SQ_d. Thus for $Q \in \mathcal{H}_{n,k}(B)$, we see that

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)} = [m-1]! \sum_{P'} \prod_{i=1}^{n} q^{s_{B_m,h}(P)_i}$$

where the second sum is over all placements P' of rooks in columns 1 through n of $SQ_{n+m-1} \supseteq B_m$ which extend to a placement $P \in \phi^{-1}_{n,B,m}(Q)$ and P is any one of these extensions of P'.

We have the following lemmas about this weighting.

Lemma 3.3. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is on the square $s_i(Q)$. Then*

$$\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)_i} = [m].$$

Proof. If $r_i(Q)$ is on $s_i(Q)$, then by definition for $P \in \phi^{-1}_{n,B,m}(Q)$ $r_i(P)$ is on one of the m lowest squares in column i not attacked by a rook to the left. The lowest square gives a contribution from column i of 1, the second lowest a contribution of $q, \ldots,$ the mth lowest a contribution of q^{m-1}. Thus we see that $\sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)_i} = [m]$. □

Lemma 3.4. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is below the square $s_i(Q)$ on the a_ith square not attacked by a rook to the left. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of q^{m-1+a_i} to each summand of* (2).

Proof. $r_i(Q)$ is on the a_ith square not attacked by a rook to the left, which is also (since $r_i(Q)$ is below $s_i(Q)$) the a_ith square not attacked by a rook to the left which does not complete a cycle. Thus we see by the definition of $\phi_{n,B,m}$ that $r_i(P)$ must be on the $(m+a_i)$th square in column i of SQ_{n+m-1} not attacked by a rook to the left. Since $r_i(Q)$ is below $s_i(Q)$ it must be on B, so by Lemma 3.2, $r_i(P)$ is on B_m. Thus $r_i(P)$ has $m-1+a_i$ uncancelled squares below it, so it contributes $m-1+a_i$ to $s_{B_m,h}(P)$ and hence a factor of q^{m-1+a_i} to each summand of (2). □

Lemma 3.5. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ on B is above the square $s_i(Q)$, and on the a_ith square not attacked by a rook to the left. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of q^{m-1+a_i-1} to each summand of* (2).

Proof. $r_i(Q)$ is on the a_ith square not attacked by a rook to the left, which is (since $r_i(Q)$ is above $s_i(Q)$) the $(a_i - 1)$th square not attacked by a rook to the left which does not complete a cycle. Thus we see by the definition of $\phi_{n,B,m}$ that $r_i(P)$ must be on the $(m + a_i - 1)$th square in column i of SQ_{n+m-1} not attacked by a rook to the left. Again by Lemma 3.2, since $r_i(Q)$ is on B $r_i(P)$ must be on B_m. Thus $r_i(P)$ has $m - 1 + a_i - 1$ uncancelled squares below it, so it contributes $m - 1 + a_i - 1$ to $s_{B_m,h}(P)$ and hence a factor of q^{m-1+a_i-1} to each summand of (2). □

Lemma 3.6. *For a fixed placement $Q \in \mathcal{H}_{n,k}(B)$, suppose a rook $r_i(Q)$ is off B. Then for every $P \in \phi^{-1}_{n,B,m}(Q)$, $r_i(P)$ contributes a factor of $q^{m-1+s_{B,h}(Q)_i}$ to each summand of* (2).

Proof. By Lemma 3.2 and its proof, we see that if $r_i(Q)$ is on (i, j) then $r_i(P)$ is on $(i, j + m - 1)$ and the number of rooks below and to the left of $r_i(Q)$ is equal to the number of rooks below and to the left of $r_i(P)$. Thus the number of squares coming from column i when calculating $s_{B_m,h}(P)$ is the same as the number of squares from column i when calculating $m - 1 + s_{B,h}(Q)$, hence such a rook contributes a factor of $q^{m-1+s_{B,h}(Q)_i}$ to each summand of (2). □

Note that for $Q \in \mathcal{P}_n(B)$ and $r_i(Q)$ not on the cycle square but on the a_ith square not attacked by a rook to the left, $a_i = s_{B,h}(Q)_i + 1$. Thus for a rook below the cycle square in column i we have that $q^{m-1+a_i} = q^{m-1+s_{B,h}(Q)_i+1}$, and for a rook on B above the cycle square in column i, $q^{m-1+a_i-1} = q^{m-1+s_{B,h}(Q)_i}$. Now we see that

$$
\begin{aligned}
A_{n,k}(m,q,B) &= \frac{1}{[m-1]!} A_{n+m-1,k}(q,B_m) = \frac{1}{[m-1]!} \sum_{P \in \mathcal{H}_{n+m-1,(n+m-1)-k}(B_m)} q^{s_{B_m,h}(P)} \\
&= \frac{1}{[m-1]!} \sum_{Q \in \mathcal{H}_{n,n-k}(B)} \left\{ \sum_{P \in \phi^{-1}_{n,B,m}(Q)} q^{s_{B_m,h}(P)} \right\} \\
&= \frac{1}{[m-1]!} [m-1]! \sum_{Q \in \mathcal{H}_{n,n-k}(B)} \left\{ \sum_{\substack{P' \text{ which extend} \\ \text{to some } P \in \phi^{-1}_{n,B,m}(Q)}} \prod_{i=1}^{n} q^{s_{B_m,h}(P)_i} \right\} \\
&= \sum_{Q \in \mathcal{H}_{n,n-k}(B)} [m]^{\operatorname{cyc}(Q)} \prod_{r_i(Q) \text{ below } s_i(Q)} q^{m-1+a_i(Q)} \\
&\quad \times \prod_{r_i(Q) \text{ above } s_i(Q) \text{ on } B} q^{m-1+a_i(Q)-1} \prod_{r_i(Q) \text{ above } s_i(Q) \text{ off } B} q^{m-1+s_{B,h}(Q)_i} \\
&= \sum_{Q \in \mathcal{H}_{n,n-k}(B)} [m]^{\operatorname{cyc}(Q)} \prod_{r_i(Q) \text{ below } s_i(Q)} q^{(m-1)+s_{B,h}(Q)_i+1} \\
&\quad \times \prod_{r_i(Q) \text{ above } s_i(Q)} q^{(m-1)+s_{B,h}(Q)_i}
\end{aligned}
$$

$$= \sum_{Q \in \mathcal{H}_{n,n-k}(B)} [m]^{\mathrm{cyc}(Q)} q^{(n-\mathrm{cyc}(Q))(m-1)+s_{B,b}(Q)+E(Q)}, \tag{3}$$

where $s_{B,b}(Q)$ is defined as the number of squares on SQ_n which neither contain a rook from P nor are cancelled, after applying the following cancellation scheme.

1. Each rook cancels all squares to the right in its row.
2. Each rook on B cancels all squares above it in its column (squares both on B and strictly above B).
3. Each rook on B which also completes a cycle cancels all squares below it in its column as well.
4. Each rook off B cancels all squares below it but above B.

Note that if we let $m = 1$ in (3), then we obtain a statistic to generate the q-hit numbers. That is,

$$A_{n,k}(q, B) = \sum_{Q \in \mathcal{H}_{n,n-k}(B)} q^{s_{B,b}(Q)+E(Q)}.$$

While this new statistic is equal to neither that of Haglund [8] nor Dworkin [3], it is a member of the family of statistics discussed by Haglund and Remmel [9, p. 479].

4. The main theorem and a corollary

We can now define

$$\tilde{A}_{n,k}(y, q, B) = \sum_{P \in \mathcal{H}_{n,n-k}(B)} [y]^{\mathrm{cyc}(P)} q^{(n-\mathrm{cyc}(P))(y-1)+s_{B,b}(P)+E(P)}$$

and prove the following.

Theorem 4.1. *For B any regular Ferrers board we have*

$$A_{n,k}(y, q, B) = \tilde{A}_{n,k}(y, q, B)$$

for $0 \leqslant k \leqslant n$.

Proof. Both of the above expressions are polynomials in the variable q^y over the field $\mathbb{Q}(q)$ of fixed degree. By the previous section, $A_{n,k}(m, q, B) = \tilde{A}_{n,k}(m, q, B)$ for any $m \in \mathbb{N}$. Thus these two polynomials have infinitely many common values, hence must be equal for all y. $\square$

A permutation statistic s is called *Mahonian* if

$$\sum_{\sigma \in S_n} q^{s(\sigma)} = [n]!.$$

We shall say that a pair (s_1, s_2) of statistics is *cycle-Mahonian* if

$$\sum_{\sigma \in S_n} [y]^{s_1(\sigma)} q^{s_2(\sigma, y)} = [y][y+1]\cdots[y+n-1].$$

Note that the statistic s_2 may depend on both σ and y. This notion generalizes that of a Mahonian statistic, since letting $y = 1$ in the definition of cycle-Mahonian gives

$$\sum_{\sigma \in S_n} q^{s_2(\sigma, 1)} = [1][2]\cdots[n] = [n]!.$$

We can associate to a permutation $\sigma \in S_n$ the placement P_σ of n rooks on SQ_n consisting of the squares $\{(i, j) \mid \sigma(i) = j\}$. We can then make any statistic stat defined for placements of n rooks on SQ_n into a permutation statistic by letting

$$\mathrm{stat}(\sigma) = \mathrm{stat}(P_\sigma).$$

In light of this definition, we have the following.

Corollary 4.2. *The pair* $(\mathrm{cyc}(-), (n - \mathrm{cyc}(-))(y - 1) + s_{B,b}(-) + E(-))$ *is cycle-Mahonian for any regular Ferrers board* $B \subseteq SQ_n$.

Proof. By definition,

$$\begin{aligned}
&\sum_{\sigma \in S_n} [y]^{\mathrm{cyc}(\sigma)} q^{(n-\mathrm{cyc}(\sigma))(y-1)+s_{B,b}(\sigma)+E(\sigma)} \\
&\quad = \sum_{\sigma \in S_n} [y]^{\mathrm{cyc}(P_\sigma)} q^{(n-\mathrm{cyc}(P_\sigma))(y-1)+s_{B,b}(P_\sigma)+E(P_\sigma)}.
\end{aligned}$$

By Theorem 4.1 we know that

$$\sum_{\sigma \in S_n} [y]^{\mathrm{cyc}(P_\sigma)} q^{(n-\mathrm{cyc}(P_\sigma))(y-1)+s_{B,b}(P_\sigma)+E(P_\sigma)} = \sum_{k=0}^{n} A_{n,k}(y, q, B).$$

Finally, it is known [7] that for any regular Ferrers board $B \subseteq SQ_n$,

$$\sum_{k=0}^{n} A_{n,k}(y, q, B) = [y][y+1]\cdots[y+n-1]. \qquad \square \tag{4}$$

5. A cycle-Euler–Mahonian pair

Recall the permutation statistics for $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$

$$\mathrm{des}(\sigma) = \bigl|\{i \mid \sigma_i > \sigma_{i+1}\}\bigr| \quad \text{and} \quad \mathrm{maj}(\sigma) = \sum_{\sigma_i > \sigma_{i+1}} i,$$

called the *number of descents* and the *major index*, respectively, of the permutation σ. The *q-Eulerian numbers* are then defined by the equation

$$E_{n,k}(q) = \sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k-1} q^{\mathrm{maj}(\sigma)}.$$

It is known [8] that $E_{n,k}(q) = A_{n,k-1}(q, \mathbb{T}_n)$, where $\mathbb{T}_n = B(1, 2, \dots, n)$ is the triangular board. Hence we obtain a q, y-version of the Eulerian numbers via the equation

$$E_{n,k}(y, q) = A_{n,k-1}(y, q, \mathbb{T}_n).$$

Now suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$. If $\sigma_{j_1} = 1$, let y_1 be the cycle $(\sigma_1\cdots\sigma_{j_1})$. If α is the smallest integer not contained in y_1, and $\sigma_{j_2} = \alpha$, let y_2 be the cycle $(\sigma_{j_1+1}\cdots\sigma_{j_2})$, etc. If the result of the above procedure is the product $y_1y_2\cdots y_p$, we will let $p = \mathrm{lrmin}(\sigma)$, called the number of *left-to-right minima* of σ. We can now define

$$\widetilde{E}_{n,k}(y, q) = \sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k-1} [y]^{\mathrm{lrmin}(\sigma)} q^{(n-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)}$$

and prove the following.

Proposition 5.1. *We have*

$$\widetilde{E}_{n,k}(y, q) = [y + k - 1]\widetilde{E}_{n-1,k}(y, q) + q^{y+k-2}[n - k + 1]\widetilde{E}_{n-1,k-1}(y, q)$$

for any $n, k \in \mathbb{N}$.

Proof. We mimic the well-known proof when $y = 1$ (that is, in the case of the regular q-Eulerian numbers $E_{n,k}(q)$). Any permutation in S_n with $k - 1$ descents can be built from one in S_{n-1} with either $k - 1$ or $k - 2$ descents in the following way.

First suppose $\sigma \in S_{n-1}$ has $k - 1$ descents, occurring at positions $i_1, i_2, \dots, i_{k-1}$. Thus $\sigma = \sigma_1\cdots\sigma_{i_1}\cdots\sigma_{i_{k-1}}\cdots\sigma_{n-1}$, where

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{i_1} > \sigma_{i_1+1} < \cdots < \sigma_{i_2} > \sigma_{i_2+1} < \cdots < \sigma_{i_{k-1}} > \sigma_{i_{k-1}+1} < \cdots < \sigma_{n-1}.$$

This permutation will contribute $[y]^{\mathrm{lrmin}(\sigma)} q^{((n-1)-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)}$ to $\widetilde{E}_{n-1,k}(y, q)$.

We can place n in any of the $k - 1$ positions of σ where a descent occurs, thereby creating a new permutation σ' in S_n which still has only $k - 1$ descents. If we place n in the $(i_1 + 1)$th position, all the descents are moved one position to the right, thus increasing maj by $k-1$. Here we see that $\mathrm{lrmin}(\sigma) = \mathrm{lrmin}(\sigma')$, since there will clearly be a number to the right of where we have placed n which is smaller than n. However, we have increased the number of letters in the permutation from $n-1$ to n. Thus

$$\begin{aligned}&[y]^{\mathrm{lrmin}(\sigma')} q^{(n-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}\\ &\quad = \left\{q^{(y-1)+(k-1)}\right\} \times [y]^{\mathrm{lrmin}(\sigma)} q^{((n-1)-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)}.\end{aligned}$$

Next we see that if we place n in the (i_2+1)th position, this time maj will increase by $k-2$, and again $\mathrm{lrmin}(\sigma') = \mathrm{lrmin}(\sigma)$ but the number of letters in the permutation increases by one. Therefore in this case, we gain a factor of $q^{(y-1)+(k-2)}$.

Continuing in this manner we proceed from left to right. Placing n in the $(i_{k-1}+1)$th position gives a factor of $q^{(y-1)+1}$, so the sum of all of these factors is $q^{y+k-2} + q^{y+k-3} + \cdots + q^{y+1} + q^{y}$. There is one last position where we can place n and not increase des, and that is the nth position. This will also not increase maj, however $\mathrm{lrmin}(\sigma')$ will now be $\mathrm{lrmin}(\sigma)+1$. We have also increased the total number of letters from $n-1$ to n, but since $\mathrm{lrmin}(\sigma') = \mathrm{lrmin}(\sigma)+1$ we have that $(n-1) - \mathrm{lrmin}(\sigma) = n - \mathrm{lrmin}(\sigma')$. Thus this last placement of n just contributes $[y]$, and summing over all positions for n which do not increase $\mathrm{des}(\sigma)$ gives $[y] + q^{y} + q^{y+1} + \cdots + q^{y+k-2}$, which is equal to $[y+k-1]$. Summing again, over all $\sigma \in S_{n-1}$ with $k-1$ descents yields the first term in the recurrence.

Now suppose $\sigma \in S_{n-1}$ has $k-2$ descents, occurring at positions $i_1, i_2, \ldots, i_{k-2}$. Thus $\sigma = \sigma_1 \cdots \sigma_{i_1} \cdots \sigma_{i_{k-2}} \cdots \sigma_{n-1}$, where

$$\sigma_1 < \sigma_2 < \cdots < \sigma_{i_1} > \sigma_{i_1+1} < \cdots < \sigma_{i_2} > \sigma_{i_2+1} < \cdots < \sigma_{i_{k-2}} > \sigma_{i_{k-2}+1} < \cdots < \sigma_{n-1}.$$

This permutation will contribute $[y]^{\mathrm{lrmin}(\sigma)} q^{((n-1)-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)}$ to $\widetilde{E}_{n-1,k-1}(y,q)$.

We can place n in any of the $n-(k-1)$ positions which will create an additional descent in our new permutation σ'. If we place n in the first position, this new descent will add 1 to maj, and it will move each of the $k-2$ descents to the right of it one position to the right, adding another $k-2$ to maj. Thus maj will increase by a total of $k-1$. As argued in the above case, $\mathrm{lrmin}(\sigma') = \mathrm{lrmin}(\sigma)$, but since we have increased the number of letters in the permutation from $n-1$ to n, $n - \mathrm{lrmin}(\sigma') = \{(n-1) - \mathrm{lrmin}(\sigma)\} + 1$. Thus we also obtain an extra q^{y-1}, and hence

$$\begin{aligned}&[y]^{\mathrm{lrmin}(\sigma')} q^{(n-\mathrm{lrmin}(\sigma'))(y-1)+\mathrm{maj}(\sigma')}\\ &\quad = \left\{q^{(y-1)+(k-1)}\right\} \times [y]^{\mathrm{lrmin}(\sigma)} q^{((n-1)-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)}.\end{aligned}$$

Continuing in this manner until the first descent at position i_1, we obtain factors of $q^{(y-1)+(k-1)}, q^{(y-1)+k}, \ldots, q^{(y-1)+k-2+i_1}$. We do not place n in the (i_1+1)th position, as this will not create a new descent. Instead, we skip over this position and move to the (i_1+2)th position. The new descent created will contribute i_1+2 to maj. Now there will be only $k-3$ descents to the right of where we have placed n, which will each be moved one position to the right increasing maj by $k-3$. As argued in the previous paragraph, we will gain a factor of $q^{(y-1)+k-3+i_1+2} = q^{(y-1)+k-1+i_i}$.

We continue the above placement scheme, skipping over positions where descents are already in σ. The last position will contribute $q^{(y-1)+n-1}$, and the sum over all positions for n in σ which increase des yields $q^{y+k-2} + q^{y+k-1} + \cdots + q^{y+n-2} = q^{y+k-2} \times \{1 + q + q^2 + \cdots + q^{n-k}\} = q^{y+k-2}[n-k+1]$. Now summing over all $\sigma \in S_{n-1}$ with $k-2$ descents yields the second term in the recurrence. □

We have the following easy lemma.

Lemma 5.2. *We have*

$$E_{n,k}(y,q) = [y+k-1]E_{n-1,k}(y,q) + q^{y+k-2}[n-k+1]E_{n-1,k-1}(y,q)$$

for $n, k \in \mathbb{N}$.

Proof. Let $B = \mathbb{T}_n$ in Lemma 2.2. □

We can now prove the following theorem.

Theorem 5.3. *For any* $n, k \in \mathbb{N}$ *we have that* $\widetilde{E}_{n,k}(y,q) = E_{n,k}(y,q)$.

Proof. It is clear that $\widetilde{E}_{1,1}(y,q) = [y]$, and it is easy to check by definition of $A_{1,1}(y,q,\mathbb{T}_1)$ that $E_{1,1}(y,q) = [y]$. Thus the $\widetilde{E}_{n,k}(y,q)$ and the $E_{n,k}(y,q)$ satisfy the same initial conditions, and they satisfy the same recurrence by Proposition 5.1 and Lemma 5.2. Hence $\widetilde{E}_{n,k}(y,q) = E_{n,k}(y,q)$ for all $n, k \in \mathbb{N}$. □

An immediate corollary of Theorem 5.3 is the following.

Proposition 5.4. *The pair* $(\mathrm{lrmin}(-), (n - \mathrm{lrmin}(-))(y-1) + \mathrm{maj}(-))$ *is cycle-Mahonian.*

Proof. By definition,

$$\sum_{\sigma \in S_n} [y]^{\mathrm{lrmin}(\sigma)} q^{(n-\mathrm{lrmin}(\sigma))(y-1)+\mathrm{maj}(\sigma)} = \sum_{k=1}^{n} \widetilde{E}_{n,k}(y,q).$$

By Theorem 5.3,

$$\sum_{k=1}^{n} \widetilde{E}_{n,k}(y,q) = \sum_{k=1}^{n} E_{n,k}(y,q).$$

Again by definition,

$$\sum_{k=1}^{n} E_{n,k}(y,q) = \sum_{k=1}^{n} A_{n,k-1}(y,q,\mathbb{T}_n),$$

which is equal to $[y][y+1]\cdots[y+n-1]$ by (4) (since $A_{n,n}(y,q,\mathbb{T}_n)=0$). $\square$

Note that if we consider the triangular board $\mathbb{T}_n \subset SQ_n$, we can bijectively associate to a permutation $\sigma=\sigma_1\sigma_2\cdots\sigma_n$ with k descents a placement of n rooks on SQ_n such that exactly k rooks lie off $\mathbb{T}_n$ in the following way (first noted in [10]). First, we find the product $y_1y_2\cdots y_p$ of cycles as was done when computing $\operatorname{lrmin}(-)$ earlier in this section. Then we place a rook on square (i,j) of SQ_n if and only if i follows j in one of the cycles y_ℓ. It is easy to verify that this placement will have exactly k rooks off $\mathbb{T}_n$, and that this procedure can be reversed. This placement is the *descent graph* of σ, which we will denote $DG(\sigma)$. Note that by Theorem 4.1 and the above discussion, we now have that

$$E_{n,k}(y,q)=\sum_{\sigma\in S_n,\,\operatorname{des}(\sigma)=k-1}[y]^{\operatorname{cyc}(DG(\sigma))}q^{(n-\operatorname{cyc}(DG(\sigma)))(y-1)+s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}.$$

We can now prove the following.

Theorem 5.5. *For any $n,k\in\mathbb{N}$*

$$\sum_{\substack{\sigma\in S_n,\,\operatorname{des}(\sigma)=k,\\ \operatorname{cyc}(DG(\sigma))=\ell}} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}=\sum_{\substack{\sigma\in S_n,\,\operatorname{des}(\sigma)=k,\\ \operatorname{lrmin}(\sigma)=\ell}} q^{\operatorname{maj}(\sigma)}.$$

Proof. We know that

$$\begin{aligned}&\sum_{\sigma\in S_n,\,\operatorname{des}(\sigma)=k}[y]^{\operatorname{cyc}(DG(\sigma))}q^{(n-\operatorname{cyc}(DG(\sigma)))(y-1)+s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}\\ &\quad=E_{n,k+1}(y,q).\end{aligned}\tag{5}$$

By Theorem 5.3, (5) is equal to $\widetilde{E}_{n,k+1}(y,q)$, where

$$\widetilde{E}_{n,k+1}(y,q)=\sum_{\sigma\in S_n,\,\operatorname{des}(\sigma)=k}[y]^{\operatorname{lrmin}(\sigma)}q^{(n-\operatorname{lrmin}(\sigma))(y-1)+\operatorname{maj}(\sigma)},$$

and hence

$$\begin{aligned}&\sum_{\sigma\in S_n,\,\operatorname{des}(\sigma)=k}[y]^{\operatorname{cyc}(DG(\sigma))}q^{(n-\operatorname{cyc}(DG(\sigma)))(y-1)+s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}\\ &\quad=\sum_{\sigma\in S_n,\,\operatorname{des}(\sigma)=k}[y]^{\operatorname{lrmin}(\sigma)}q^{(n-\operatorname{lrmin}(\sigma))(y-1)+\operatorname{maj}(\sigma)}.\end{aligned}\tag{6}$$

If we let $z=[y]q^{-(y-1)}$ in (6), then we have that

$$q^{n(y-1)} \sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} z^{\mathrm{cyc}(DG(\sigma))} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}$$
$$= q^{n(y-1)} \sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} z^{\mathrm{lrmin}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

Thus

$$\sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} z^{\mathrm{cyc}(DG(\sigma))} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))}$$

and

$$\sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} z^{\mathrm{lrmin}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

are equal polynomials in the variable z over $\mathbb{N}[q]$, and hence equal powers of z must have equal coefficients. In particular the coefficient of z^{ℓ} in each must be equal. That is

$$\sum_{\substack{\sigma\in S_n,\, \mathrm{des}(\sigma)=k\\ \mathrm{cyc}(DG(\sigma))=\ell}} q^{s_{\mathbb{T}_n,b}(DG(\sigma))+E(DG(\sigma))} = \sum_{\substack{\sigma\in S_n,\, \mathrm{des}(\sigma)=k\\ \mathrm{lrmin}(\sigma)=q^{\mathrm{maj}(\sigma)}}}$$

as desired. □

Recall that a permutation statistic s on S_n is called *Euler–Mahonian* if the pairs (des, s) and $(\mathrm{des}, \mathrm{maj})$ have the same distribution on S_n, that is,

$$\sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} q^{s(\sigma)} = \sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} q^{\mathrm{maj}(\sigma)}$$

for all values of k. Theorem 5.5 leads us to define the following generalization. We will say a pair of permutation statistics $(s_1(-), s_2(-, y))$ is *cycle-Euler–Mahonian* if it is cycle-Mahonian as defined in Section 4, and

$$\sum_{\substack{\sigma\in S_n,\, \mathrm{des}(\sigma)=k,\\ s_1(\sigma)=\ell}} q^{s_2(\sigma,1)} = \sum_{\substack{\sigma\in S_n,\, \mathrm{des}(\sigma)=k,\\ \mathrm{lrmin}(\sigma)=\ell}} q^{\mathrm{maj}(\sigma)}. \tag{7}$$

This definition generalizes that of Euler–Mahonian, because if $(s_1(-), s_2(-, y))$ satisfies (7) then

$$\sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} q^{s_2(\sigma,1)} = \sum_{\ell} \Biggl\{ \sum_{\substack{\sigma\in S_n,\, \mathrm{des}(\sigma)=k,\\ s_1(\sigma)=\ell}} q^{s_2(\sigma,1)} \Biggr\} = \sum_{\ell} \Biggl\{ \sum_{\substack{\sigma\in S_n,\, \mathrm{des}(\sigma)=k,\\ \mathrm{lrmin}(\sigma)=\ell}} q^{\mathrm{maj}(\sigma)} \Biggr\}$$
$$= \sum_{\sigma\in S_n,\, \mathrm{des}(\sigma)=k} q^{\mathrm{maj}(\sigma)}.$$

Thus if $(s_1(-), s_2(-, y))$ is cycle-Euler–Mahonian, this implies that $s_2(-, 1)$ is Euler–Mahonian.

By Corollary 4.2, $(\mathrm{cyc}(DG(-)), (n - \mathrm{cyc}(DG(-)))(y - 1) + s_{\mathbb{T}_n,b}(DG(-)) + E(DG(-)))$ is cycle-Mahonian, and by Theorem 5.5 we see that

$$\bigl(\mathrm{des}(-), \mathrm{cyc}\bigl(DG(-)\bigr), s_{\mathbb{T}_n,b}\bigl(DG(-)\bigr) + E\bigl(DG(-)\bigr)\bigr)$$

and

$$\bigl(\mathrm{des}(-), \mathrm{lrmin}(-), \mathrm{maj}(-)\bigr)$$

have the same distribution. Thus

$$\bigl(\mathrm{cyc}\bigl(DG(-)\bigr), \bigl(n - \mathrm{cyc}\bigl(DG(-)\bigr)\bigr)(y-1) + s_{\mathbb{T}_n,b}\bigl(DG(-)\bigr) + E\bigl(DG(-)\bigr)\bigr)$$

is an example of a cycle-Euler–Mahonian pair of statistics on S_n.

References

[1] F. Chung, R. Graham, On the cover polynomial of a digraph, J. Combin. Theory Ser. B 65 (1995) 273–290.
[2] M. Dworkin, Factorization of the cover polynomial, J. Combin. Theory Ser. B 71 (1997) 17–53.
[3] M. Dworkin, An interpretation for Garsia and Remmel's q-hit numbers, J. Combin. Theory Ser. A 81 (1998) 149–175.
[4] E. Ehrenborg, J. Haglund, M. Readdy, Colored juggling patterns and weighted rook placements, Unpublished manuscript.
[5] A.M. Garsia, J.B. Remmel, q-Counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41 (1986) 246–275.
[6] I.M. Gessel, Generalized rook polynomials and orthogonal polynomials, in: D. Stanton (Ed.), q-Series and Partitions, in: IMA Vol. Math. Appl., Springer-Verlag, 1989.
[7] J. Haglund, Rook theory and hypergeometric series, Adv. in Appl. Math. 17 (1996) 408–459.
[8] J. Haglund, q-Rook polynomials and matrices over finite fields, Adv. in Appl. Math. 20 (1998) 450–487.
[9] J. Haglund, J.B. Remmel, Rook theory for perfect matchings, Adv. in Appl. Math. 27 (2001) 438–481.
[10] I. Kaplansky, J. Riordan, The problem of the rooks and its applications, Duke Math. J. 13 (1946) 259–268.